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**Test 3**
Spring 2007
MATH 121 Section 02
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**Directions :** You have 50 minutes to complete all 6 problems on this exam. There are a possible 100 points to be earned. You may not use your book, notes, or any graphing/programmable calculator. Please be sure to show all pertinent work. *An incorrect answer with no work will receive no credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1. (10 points)
If $z^2 = x^2 + y^2$, $dx/dt = 2$, and $dy/dt = 3$, find $dz/dt$ when $x = 5$ and $y = 12$.

**Solution :** We first compute an implicit derivative with respect to the variable $t$.

$$\begin{aligned} \frac{d}{dt}\left(z^2\right) &= \frac{d}{dt}\left(x^2 + y^2\right) \\ \Rightarrow 2z\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt}. \end{aligned}$$

Then we solve for $dz/dt$.

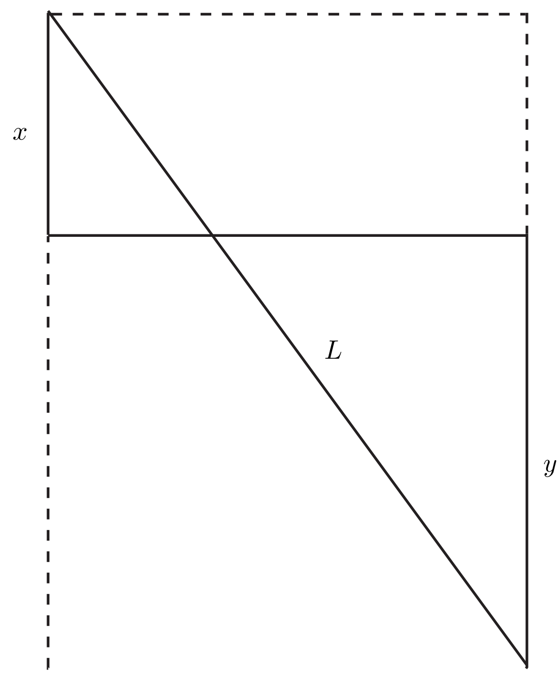
$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}$$

Using the fact that $z^2 = x^2 + y^2$, $x = 5$, and $y = 12$ we find that $z^2 = 25 + 144 = 169$, so $z = 13$. We now have all the data we need.

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} = \frac{(5)(2) + (12)(3)}{13} = \frac{46}{13}.$$

2. (20 points)
   A man starts walking north at 4 ft/s from a point $P$. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of $P$. At what rate are the people moving apart 15 min after the woman starts walking?

   **Solution :** First we sketch a picture.




From here we see that we have a right triangle whose side lengths are $L$, 500, and $x + y$. These quantities are related by the pythagorean theorem in the following way:

$$L^2 = 500^2 + (x + y)^2$$

We wish to know the rate of change of $L$ so we compute an implicit derivative with respect to $t$.

$$2L\frac{dL}{dt} = 2(x + y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

Solving for the desired quantity gives us

$$\frac{dL}{dt} = \frac{1}{L}(x + y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) \qquad (1)$$

The problem tells us that the man is walking north at 4 ft/s, so $dx/dt = 4$. Similarly, $dy/dt = 5$. When the woman has been walking for 15 minutes at 5 ft/s she has travelled a distance $y = 5(15)(60) = 4500$ feet. After the woman has been walking for 15 minutes the man has been walking for 20 minutes at a rate of 4 ft/s. Therefore, the distance he has walked is $x = 4(20)(60) = 4800$ feet. Finally, if $x = 4800$ and $y = 4500$, then

$$L = \sqrt{500^2 + (4800 + 4500)^2} = \sqrt{86740000}$$

Putting all of this data into equation (1) we have the solution

$$\begin{aligned}
\frac{dL}{dt} &= \frac{1}{\sqrt{86740000}}(4800 + 4500)\,(4 + 5) \\
&= \frac{83700}{\sqrt{86740000}} \\
&\approx 8.98702084641\text{ft/s.}
\end{aligned}$$

3. (20 points)
   Use differentials (or, equivalently, a linear approximation) to estimate $(8.06)^{2/3}$.
   **Solution :** Let $f(x) = x^{2/3}$. We are interested in computing an approximation of $f(8.06)$. To do this we need to compute a tangent line at some point on the curve $y = f(x)$ that is close to the point $(8.06, f(8.06))$. The equation for a tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is given by $\ell(x) = f'(a)(x - a) + f(a)$. From this we see that we will want to choose a value of $a$ so that we can compute $f(a)$ and $f'(a)$. Also, the closer $a$ is to 8.06 the better. Well, $f'(x) = \frac{2}{3}x^{-1/3}$ and I can readily compute $f(8) = 4$ and $f'(8) = \frac{1}{3}$. The equation of the tangent line is

$$\ell(x) = \frac{1}{3}(x - 8) + 4$$

   and

$$f(8.06) \approx \ell(8.06) = \frac{1}{3}(0.06) + 4 = 4.02.$$

   For those keeping score at home, the calculator on my macbook says that $8.06^{2/3} = 4.01997508295$. Not too shabby.

4. (10 points)
   The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm.

   (a) Use differentials to estimate the maximum error in the calculated surface area. ($A = 4\pi r^2$)

   (b) Use differentials to estimate the maximum error in the calculated volume. ($V = \frac{4}{3}\pi r^3$)

   **Solution :** For each of these it will be important to write the radius $r$ as a function of the circumference $C$. Mercifully, there is an equation which does this for us since the circumference of a circle is equal to two pi times the radius. That is $C = 2\pi r$, so $r = C/(2\pi)$.

   (a) First we rewrite the surface area with respect to the circumference as $A = 4\pi(C/(2\pi))^2 = C^2/\pi$. Then, we compute a derivative with respect to $C$.
   $$\frac{dA}{dC} = \frac{2C}{\pi}.$$
   Using the fact that the error term we want an approximation for is $dA$ and the area of the circumference measure is given as 0.5 we have
   $$dA = \frac{2C}{\pi}dC = \frac{2(84)}{\pi}(0.5) = \frac{84}{\pi}.$$

   (b) This part is very similar to the previous part. We first rewrite the volume as a function of $C$. Indeed, $V = \frac{4}{3}\pi(C/(2\pi))^3 = C^3/(6\pi^2)$. Computing a derivative with respect to $C$ gies us
   $$\frac{dV}{dC} = \frac{C^2}{2\pi^2}.$$
   Hence
   $$dV = \frac{C^2}{2\pi^2}dC = \frac{84^2}{2\pi^2}(0.5) = \frac{1764}{\pi^2}.$$

5. (20 points)
Find the absolute maximum and absolute minimum values of

$$f(x) = \frac{x}{x^2+1}$$

on the interval $[0, 2]$.

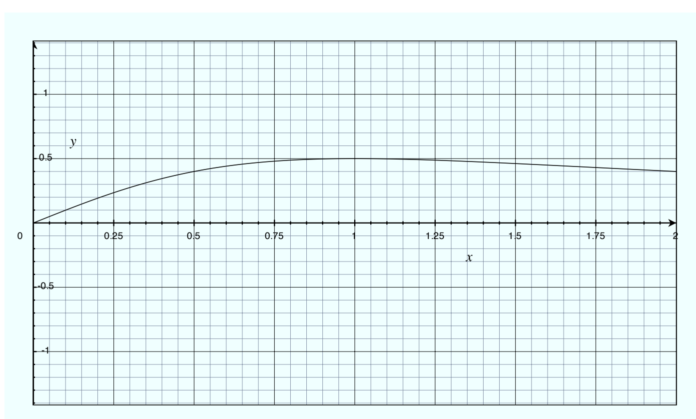
**Solution :** The given function is a ratio of polynomials and so it is continuous on its domain. Since the denominator is $x^2 + 1$, it is never zero (indeed $x^2 + 1 \geq 1$ for all real numbers $x$), so the function is continuous on the interval $(-\infty, \infty)$. It follows that on the closed interval $[0, 2]$ the function must achieve both an absolute maximum and an absolute minimum. Moreover, these points at which the extrema are obtained must either be the endpoints ($x = 0$ or $x = 2$) or critical points (points for which $f'(x) = 0$ or $f'(x)$ is undefined). The first step is to compute a derivative using the quotient rule

$$f'(x) = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

This shows us that the only critical points for the function are $x = 1$ and $x = -1$, but the only one of these we care about is $x = 1$ since this is the only point in the interval $[0, 2]$. Now we just evaluate the function at all of our candidates and compare.

$$f(0) = 0 \text{ (MIN!)}$$
$$f(1) = \frac{1}{2} \text{ (MAX!)}$$
$$f(2) = \frac{2}{5}$$

Below is a plot of the graph $y = f(x)$ on the interval $[0, 2]$.



6. (20 points)
Show that the equation $1 + 2x + x^3 + 4x^5 = 0$ has exactly one real root. (Please write complete sentences.)

**Solution :** The function $f(x) = 1 + 2x + x^3 + 4x^5$ is a polynomial, hence it is both continuous and differentiable. Observe that $f(-1) = -6$ and $f(1) = 8$. By the intermediate value theorem, there must exist at least one value $c$ in between $-1$ and 1 so that $f(c) = 0$. To see that there is only one solution we compute a derivative. Indeed, $f'(x) = 2+3x^2+20x^4$. Notice that each term in $f'(x)$ is always positive, hence $f'(x)$ is always positive. If there were to be more than one solution, then Rolle's theorem would require there to be a point at which the derivative vanishes (is equal to zero). But this is not possible since $f'(x) = 2 + 3x^2 + 20x^4 \geq 2$ for all $x$. It follows that there can be only one solution.