Section 8.4 #31

We are asked to compute the following integral

$$\int \frac{1}{x^3-1}dx.$$

As we saw in class, we use partial fractions to write

$$\int \frac{1}{x^3-1}dx = \int \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}dx$$

where we solve for for A, B, and C by recombining the fraction above to get

$$A(x^2+x+1)+(Bx+C)(x-1)=1$$

which is the same as

$$(A+B)x^2+(A-B+C)x+(A-C)=1.$$

From this we see that we want to solve the system of equations

$$A+B=0$$
$$A-B+C=0$$
$$A-C=1.$$

As we saw from class, this has the solution $A = 1/3$, $B = -1/3$, and $C = -2/3$. So our integral now becomes:

$$\int \frac{1/3}{x-1}+\frac{-1/3x-2/3}{x^2+x+1}dx = \frac{1}{3}\int \frac{1}{x-1}dx-\frac{1}{3}\int \frac{x}{x^2+x+1}dx-\frac{2}{3}\int \frac{1}{x^2+x+1}dx.$$

The first integral is not really a problem . We let $u = x-1$ and then $du = dx$ so we have

$$\frac{1}{3}\int \frac{1}{x-1}dx = \frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\ln|u| = \frac{1}{3}\ln|x-1|.$$

The next integral seems to be a candidate for u-substitution as well with $u = x^2+x+1$. But with that substitution $du = 2x+1dx$ and all we have is an xdx so this isn't quite going to work. If we had $x+1/2$ in the numerator instead of just x all would be well as we could replace $x+1/2dx$ with $du/2$. We can make this happen by fiddling with the last two integrals as follows

$$\int \frac{1}{x^3-1}dx = \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x}{x^2+x+1}dx - \frac{2}{3}\int \frac{1}{x^2+x+1}dx$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\left[\int \frac{x}{x^2+x+1}dx + \int \frac{2}{x^2+x+1}dx\right]$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\left[\int \frac{x+1/2}{x^2+x+1}dx + \int \frac{3/2}{x^2+x+1}dx\right]$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+1/2}{x^2+x+1}dx - \frac{1}{2}\int \frac{1}{x^2+x+1}dx$$

Now making the previously mentioned u-substitution with $u = x^2+x+1$ and $du/2 = x+1/2dx$ the middle integral becomes

$$-\frac{1}{3}\int \frac{x+1/2}{x^2+x+1}dx = -\frac{1}{3}\int \frac{1/2}{u}du = -\frac{1}{6}\ln|u| = -\frac{1}{6}\ln|x^2+x+1| = -\frac{1}{6}\ln(x^2+x+1).$$

We could drop the absolute value since x^2+x+1 is always positive. Okay, just to recap we now know that

$$\int \frac{1}{x^3-1}dx = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{2}\int \frac{1}{x^2+x+1}dx$$

The last integral requires a little more of a trick. If the denominator was a sum of squares (like a^2+x^2) we know that the antiderivative will involve arctan. We can force this to be the case by completing the square as follows

$$-\frac{1}{2}\int \frac{1}{x^2+x+1}dx = -\frac{1}{2}\int \frac{1}{(x+1/2)^2+3/4}dx$$

$$= -\frac{1}{2}\int \frac{1}{u^2+3/4}du \text{ (we let } u = x+1/2)$$

$$= -\frac{1}{2}\int \frac{1}{u^2+\left(\sqrt{3}/2\right)^2}du$$

$$= -\frac{1}{2}\left(\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\arctan\left(\frac{u}{\left(\frac{\sqrt{3}}{2}\right)}\right)\right)$$

$$= -\frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$$

Now we are done,

$$\int \frac{1}{x^3-1} dx = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$