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Test 3
Spring 2007
MATH 122 Section 01
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Directions : You have 50 minutes to complete 4 of the 5 problems on this exam. Below please list the four (and only four) problems that you would like for me to grade. There are a possible 100 points to be earned. Please be sure to show all pertinent work. *An correct answer with no work will receive little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

Please list the four problems you wish graded here.

________ ________ ________ ________

1. Evaluate the integrals:

a) $\displaystyle\int \frac{dx}{\sqrt{x^2+16}}$ b) $\displaystyle\int_0^{2/3} x^3\sqrt{4-9x^2}\,dx$

Solution :

a) This is a prime candidate for a trig substitution with $x = 4\tan(\theta)$.

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+16}} &= \int \frac{4\sec^2(\theta)}{\sqrt{16\tan^2(\theta)+16}}d\theta \\ &= \int \frac{4\sec^2(\theta)}{4\sqrt{\tan^2(\theta)+1}}d\theta \\ &= \int \frac{\sec^2(\theta)}{\sqrt{\sec^2(\theta)}}d\theta \\ &= \int \sec(\theta)d\theta \\ &= \ln|\sec(\theta)+\tan(\theta)| + C \\ &= \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C\end{aligned}$$

b) We start by manipulating the integrand so that we may apply a trigonometric substitution.

$$\begin{aligned}\int_0^{\frac{2}{3}} x^3\sqrt{4-9x^2}\,dx &= \int_0^{\frac{2}{3}} x^3\sqrt{9\left(\frac{4}{9}-x^2\right)}dx \\ &= \int_0^{\frac{2}{3}} 3x^3\sqrt{\left(\frac{2}{3}\right)^2 - x^2}\,dx\end{aligned}$$

Now we make the substitution $x = \frac{2}{3}\sin(\theta)$ and $dx = \frac{2}{3}\cos(\theta)d\theta$.

$$\begin{aligned}&\int_0^{\frac{2}{3}} 3x^3\sqrt{\left(\frac{2}{3}\right)^2 - x^2}\,dx \\ &= \int_0^{\frac{\pi}{2}} 3\left(\frac{2}{3}\sin(\theta)\right)^3\sqrt{\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2\sin^2(\theta)}\left(\frac{2}{3}\cos(\theta)\right)d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{32}{81}\sin^3(\theta)\cos(\theta)\sqrt{1-\sin^2(\theta)}\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{32}{81}\sin^3(\theta)\cos(\theta)\sqrt{\cos^2(\theta)}\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{32}{81}\sin^3(\theta)\cos^2(\theta)d\theta\end{aligned}$$

Now we need to prepare to make a u-substitution. So, we will peel of one of the $\sin(\theta)$'s and let $u = \cos(\theta)$.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{32}{81}\sin^3(\theta)\cos^2(\theta)d\theta &= \int_0^{\frac{\pi}{2}} \frac{32}{81}\sin(\theta)\left(1-\cos^2(\theta)\right)\cos^2(\theta)d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{32}{81}\sin(\theta)\left(\cos^2(\theta)-\cos^4(\theta)\right)d\theta \\
&= \int_1^0 -\frac{32}{81}(u^2-u^4)du \\
&= \int_0^1 \frac{32}{81}(u^2-u^4)du \\
&= \frac{32}{81}\left(\frac{u^3}{3}-\frac{u^5}{5}\right)\Bigg|_0^1 \\
&= \frac{32}{81}\left(\frac{1}{3}-\frac{1}{5}\right) \\
&= \frac{32}{81}\left(\frac{2}{15}\right) \\
&= \frac{64}{1215}
\end{aligned}$$

2. Write out the form of the partial fraction decomposition of the function.
Do not determine the numerical value of the coefficients.

a) $\dfrac{2x}{(x+3)(3x+1)}$ b) $\dfrac{1}{x^3+2x^2+x}$

Solution :

a)

$$\frac{2x}{(x+3)(3x+1)} = \frac{A}{x+3} + \frac{B}{3x+1}$$

b)

$$\frac{1}{x^3+2x^2+x} = \frac{1}{x(x^2+2x+1)} = \frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

3. Evaluate the integral:

$$\int \frac{5x^2 + 3x - 2}{x^3 + 2x^2} dx$$

Solution :

$$\int \frac{5x^2 + 3x - 2}{x^3 + 2x^2} dx = \int \frac{5x^2 + 3x - 2}{x^2(x+2)} dx = \int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} dx$$

Now we need to find A, B, and C. Getting a common denominator we can add up the fractions on the right as

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} = \frac{Ax(x+2) + B(x+2) + Cx^2}{x^2(x+2)} = \frac{(A+C)x^2 + (2A+B)x + 2B}{x^2(x+2)}$$

Since we know $(A+C)x^2 + (2A+B)x + 2B = 5x^2 + 3x - 2$ we have a system of equations to solve.

$$\begin{aligned} A + C &= 5 \\ 2A + B &= 3 \\ 2B &= -2 \end{aligned}$$

This has as a solution $A = 2$, $B = -1$, and $C = 3$. Now we can integrate!

$$\begin{aligned} \int \frac{5x^2 + 3x - 2}{x^3 + 2x^2} dx &= \int \frac{2}{x} dx + \int \frac{-1}{x^2} dx + \int \frac{3}{x+2} dx \\ &= 2 \ln |x| + \frac{1}{x} + 3 \ln |x+2| + C \end{aligned}$$

4. Find the approximations T_6 and S_6 for $\int_0^1 e^x dx$ and the corresponding errors E_T and E_S. How large do we have to choose n so that the approximations T_n, M_n, and S_n to the integral $\int_0^1 e^x dx$ are accurate to within 0.00001?

Solution : The values we want are

$$
\begin{aligned}
T_6 &= \frac{1}{12}\left(f(0) + 2f\left(\frac{1}{6}\right) + 2f\left(\frac{2}{6}\right) + 2f\left(\frac{3}{6}\right) + 2f\left(\frac{4}{6}\right) + 2f\left(\frac{5}{6}\right) + f(1)\right) \\
&= \frac{1}{12}\left(e^0 + 2e^{1/6} + 2e^{2/6} + 2e^{3/6} + 2e^{4/6} + 2e^{5/6} + e^1\right),
\end{aligned}
$$

$$
\begin{aligned}
S_6 &= \frac{1}{18}\left(f(0) + 4f\left(\frac{1}{6}\right) + 2f\left(\frac{2}{6}\right) + 4f\left(\frac{3}{6}\right) + 2f\left(\frac{4}{6}\right) + 4f\left(\frac{5}{6}\right) + f(1)\right) \\
&= \frac{1}{18}\left(e^0 + 4e^{1/6} + 2e^{2/6} + 4e^{3/6} + 2e^{4/6} + 4e^{5/6} + e^1\right),
\end{aligned}
$$

$$|E_T| \le \frac{K(1-0)^3}{12(6^2)} \le \frac{e}{432},$$

and

$$|E_S| \le \frac{K(1-0)^5}{180n^4} \le \frac{e}{233280},$$

For the second part we want to find the least n so that

$$
\begin{aligned}
|E_T| &\le \frac{e}{12n_T^2} \le 0.00001 \\
|E_M| &\le \frac{e}{24n_M^2} \le 0.00001 \\
|E_S| &\le \frac{e}{180n_S^4} \le 0.00001
\end{aligned}
$$

These have the following solutions for n respectively

$$
\begin{aligned}
n_T &\ge \sqrt{\frac{e}{12(0.00001)}} \approx 150.50697 \\
n_M &\ge \sqrt{\frac{e}{24(0.00001)}} \approx 106.42450 \\
n_S &\ge \left[\frac{e}{180(0.00001)}\right]^{1/4} \approx 6.23384
\end{aligned}
$$

So n must be at least 151, 107, and 7 respectively.

5. Evaluate the integral:

$$\int \frac{\sqrt{x^2-9}}{x^3} dx$$

[Just FYI $\sin 2\theta = 2\sin\theta\cos\theta$.] **Solution :** We'll start things off with a substitution $x = 3\sec(\theta)$ and $dx = 3\sec(\theta)\tan(\theta)d\theta$.

$$\int \frac{\sqrt{x^2-9}}{x^3} dx = \int \frac{3\sec(\theta)\tan(\theta)\sqrt{9\sec^2(\theta)-9}}{27\sec^3(\theta)} d\theta$$

$$= \int \frac{9\sec(\theta)\tan(\theta)\sqrt{\sec^2(\theta)-1}}{27\sec^3(\theta)} d\theta$$

$$= \int \frac{9\sec(\theta)\tan(\theta)\sqrt{\tan^2(\theta)}}{27\sec^3(\theta)} d\theta$$

$$= \int \frac{9\sec(\theta)\tan^2(\theta)}{27\sec^3(\theta)} d\theta$$

$$= \int \frac{\tan^2(\theta)}{3\sec^2(\theta)} d\theta$$

$$= \int \frac{\sec^2(\theta)-1}{3\sec^2(\theta)} d\theta$$

$$= \frac{1}{3}\int d\theta - \frac{1}{3}\int \cos^2(\theta)d\theta$$

$$= \frac{\theta}{3} - \frac{1}{3}\int \frac{1+\cos(2\theta)}{2} d\theta$$

$$= \frac{\theta}{3} - \frac{1}{6}\int 1+\cos(2\theta)d\theta$$

$$= \frac{\theta}{3} - \frac{\theta}{6} - \frac{\sin(2\theta)}{12} + C$$

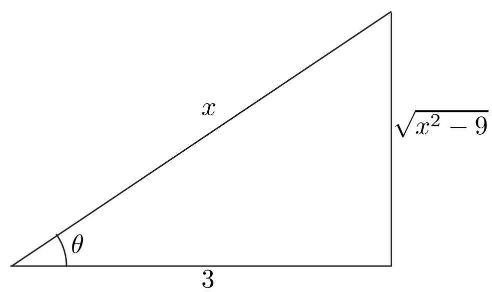
$$= \frac{2\theta - \sin(2\theta)}{12} + C$$

Now we use the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.

$$\frac{2\theta - \sin(2\theta)}{12} + C = \frac{2\theta - 2\sin(\theta)\cos(\theta)}{12} + C$$

$$= \frac{\theta}{6} - \frac{1}{6}\sin(\theta)\cos(\theta) + C$$

We made the substitution $x = 3\sec(\theta)$. We can represent this datum on a triangle as follows: The figure shows us that $\sin(\theta) = \sqrt{x^2 - 9}/x$ and



$\cos(\theta) = 3/x$. This completes the problem.

$$\int \frac{\sqrt{x^2 - 9}}{x^3} dx = \frac{\theta}{6} - \frac{1}{6}\sin(\theta)\cos(\theta) + C$$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\left(\frac{\sqrt{x^2 - 9}}{x}\right)\left(\frac{3}{x}\right) + C$$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2 - 9}}{2x^2} + C$$