

Name: Dr. Mullikin

Test 4
Spring 2007
MATH 122 Section 01
March 26, 2007

Directions : You have 50 minutes to complete all 4 problems on this exam. There are a possible 100 points to be earned. Please be sure to show all pertinent work. *An correct answer with no work will receive little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1. (25 points)

Use the arc length formula to find the length of the curve $y = 2 - 3x$, $-2 \leq x \leq 1$. Check your answer by noting that the curve is a line segment and calculating its length by the distance formula.

Solution :

$$L = \int_{-2}^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{-2}^1 \sqrt{1 + (-3)^2} dx = \int_{-2}^1 \sqrt{10} dx = 3\sqrt{10}.$$

Now we check using the distance formula.

$$L = \sqrt{(-2 - 1)^2 + ((2 - 3(-2)) - (2 - 3(1)))^2} = \sqrt{9 + 81} = \sqrt{90} = 3\sqrt{10}.$$

2. (25 points)

Find the area of the surface obtained by rotating the curve $y = \sqrt{x}$, $4 \leq x \leq 9$ about the x -axis.

Solution : The surface area is given by

$$\begin{aligned} A &= \int_4^9 2\pi\sqrt{x} \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} dx \\ &= \int_4^9 2\pi\sqrt{x} \sqrt{\frac{4x+1}{4x}} dx \\ &= \int_4^9 2\pi\sqrt{x} \frac{\sqrt{4x+1}}{\sqrt{4x}} dx \\ &= \int_4^9 \pi\sqrt{4x+1} dx \\ &= \int_{17}^{37} \frac{\pi}{4} \sqrt{u} du && \text{(we let } u = 4x + 1) \\ &= \frac{\pi}{4} \left(\frac{2}{3}u^{3/2}\right) \Big|_{17}^{37} \\ &= \frac{\pi}{6} \left(37^{3/2} - 17^{3/2}\right). \end{aligned}$$

3. (25 points)

Use the theorem of Pappus to find the volume of a cone with height h and base radius r .

Solution : We set up the problem according to the diagram below.

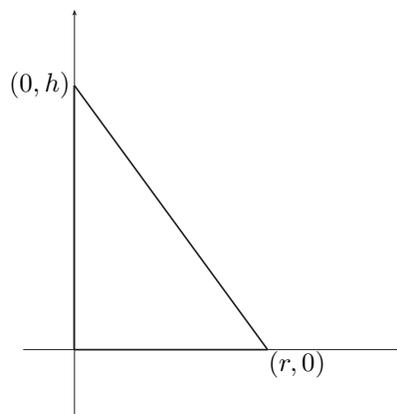


Figure 1: We will obtain a cone of base radius r and height h by rotating the triangular region shown about the y -axis

Pappus says that we need only find the x -coordinate of the centroid (\bar{x}, \bar{y}) and then the total volume will be $V = 2\pi\bar{x}A$ where A is the area of the triangle, that is $A = \frac{rh}{2}$. It remains to compute \bar{x} . Indeed,

$$\begin{aligned}\bar{x} &= \frac{1}{A} \int_0^r xf(x)dx \\ &= \frac{2}{rh} \int_0^r x \left(-\frac{h}{r}(x-r) \right) dx \\ &= \frac{2}{r^2} \int_0^r -x^2 + rxdx \\ &= \frac{2}{r^2} \left(-\frac{x^3}{3} + \frac{rx^2}{2} \right) \Big|_0^r \\ &= \frac{2}{r^2} \left(-\frac{r^3}{3} + \frac{r^3}{2} \right) \\ &= \frac{2}{r^2} \left(\frac{r^3}{6} \right) \\ &= \frac{r}{3}\end{aligned}$$

The volume is then

$$V = 2\pi\bar{x}A = 2\pi \left(\frac{r}{3} \right) \left(\frac{rh}{2} \right) = \frac{\pi r^2 h}{3}.$$

4. (25 points)

Find a formula for the general term a_n of the sequence $\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \dots\}$, assuming that the pattern of the first few terms continues. If the sequence converges, what is the limit?

Solution :

$$a_n = \left(\frac{-2}{3}\right)^{n-1}.$$

This sequence does converge and it converges to 0.