MATH 462
Final Examination

You will have until Wednesday, May 7th 11:00am to complete the following problems. You may use your calculus book, Shifrin's book, your brain, and my brain. Please do not use any other books or anyone else's brain. You must work any 5 of the following 9 problems. Please only turn in 5. If you turn in more than 5, I will grade the first 5.

1. Find the area inside the hypocycloid $x^{2/3} + y^{2/3} = 1$. You should parametrize by $g(t) = \begin{pmatrix} \cos^3(t) \\ \sin^3(t) \end{pmatrix}$.

2. Define $\star : \mathcal{A}^1(\mathbb{R}^2) \longrightarrow \mathcal{A}^1(\mathbb{R}^2)$ by $\star dx = dy$, and $\star dy = -dx$, extending by linearity. If f is a smooth function, show that
$$d \star (df) = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx \wedge dy.$$

3. An ant finds himself in the xy-plane in the presence of the force field $\mathbf{F} = \begin{pmatrix} y^3 + x^2 y \\ 2x^2 - 6xy \end{pmatrix}$. Around what simple closed curve beginning and ending at the origin should he travel counterclockwise (once) in order to maximize the work done on him by $\mathbf{F}$?

4. Find the surface area of the torus given parametrically by $g : (0, 2\pi) \times (0, 2\pi) \longrightarrow \mathbb{R}^3$,
$$g \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (a + b\cos v)\cos u \\ (a + b\cos v)\sin u \\ b \sin v \end{pmatrix}.$$

5. Find the flux of the vector field $\mathbf{F} = \begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix}$ outward across the surface S where

   (a) S is the sphere of radius a centered at the origin.

   (b) S is the upper hemisphere of radius a centered at the origin.

   (In each case the outward-pointing normal points away from the origin.)

6. Calculate the flux of the vector field $\mathbf{F} = \begin{pmatrix} xz \\ yz \\ x^2 + y^2 \end{pmatrix}$ outward across the surface of the paraboloid S given by $z = 4 - x^2 - y^2$, $z \geq 0$ (with outward-pointing normal having positive $\mathbf{e}_3$-component.)

7. Let C be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $2x + 3y - z = 1$, oriented counterclockwise as viewed from high above the xy-plane. Evaluate
$$\int_C y\,dx - 2z\,dy + x\,dz$$
directly ***and*** by applying Stokes's Theorem.

8. Let $\omega = y^2\,dy \wedge dz + x^2\,dz \wedge dx + z^2\,dx \wedge dy$, and let M be the solid paraboloid $0 \leq z \leq 1 - x^2 - y^2$. Evaluate $\int_{\partial M} \omega$ directly ***and*** by applying Stokes's Theorem.

9. Suppose X is a compact, oriented k-dimensional manifold and $\mathbf{f} : X \longrightarrow Y$, $\mathbf{g} : X \longrightarrow Y$ are homotopic maps. Show for any closed k-form ω on Y, we have
$$\int_X \mathbf{f}^*\omega = \int_X \mathbf{g}^*\omega.$$