MATH 462
Homework 2

Here is a collection of problems that should help facilitate your understanding of differential forms. There are two sections. The first is a collection of computation exercises which you should work to become comfortable with the notation and processes. The suggested problems will not be turned in, though I will be happy to answer any questions you have abut them or the required problems. The latter set of problems *will* be turned in (on Tuesday March 18th) and graded. As always, you are encouraged to work in groups, but please make sure that you write up your own solutions.

1 Suggested Problems

1. Shifrin: pg. 345 #4.
2. Shifrin: pg. 345 #6.
3. Shifrin: pg. 345 #7,
4. Shifrin: pg. 345 #8.
5. Shifrin: pg. 345 #11.

2 Required Problems

1. Shifrin: pg. 334-335 #2.

2. Let $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$. Sketch the two vectors in $\mathbb{R}^3$ and sketch the corresponding parallelogram (with vertices at the origin, the head of **a**. the head of **b**, and the head of $\mathbf{a} + \mathbf{b}$). In three separate graphs, sketch the projection of the parallelogram in the xy, xz, and yz planes and find the area of each of the three parallelograms.

3. Using **a** and **b** from the previous exercise, compute $d\mathbf{x}_{(1,2)}$, $d\mathbf{x}_{(1,3)}$, and $d\mathbf{x}_{(2,3)}$. What, if anything, do you notice?

4. Let $\mathbf{n} \in \mathbb{R}^3$ be a unit vector, and let **v** and **w** be orthogonal to **n**. Let

$$\phi = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy.$$

   Prove that $\phi(\mathbf{v}, \mathbf{w})$ is equal to the signed area of the parallelogram spanned by **v** and **w** (the sign being determined by whether **n**, **v**, **w** form a right handed system for $\mathbb{R}^3$).

5. (a) Suppose $\omega \in \Lambda^k (\mathbb{R}^n)^*$ and k is odd. Prove that $\omega \wedge \omega = 0$.
   (b) Give an example to show that the result of part (a) need not hold when k is even.

6. Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Show that $dx(\mathbf{v} \times \mathbf{w}) = dy \wedge dz(\mathbf{v}, \mathbf{w}$, $dy(\mathbf{v} \times \mathbf{w}) = dz \wedge dx(\mathbf{v}, \mathbf{w}$, and $dz(\mathbf{v} \times \mathbf{w}) = dx \wedge dy(\mathbf{v}, \mathbf{w}$.

7. Let $\mathbf{g} : (0, \infty) \times (0, \pi) \times (0, 2\pi) \longrightarrow \mathbb{R}^3$ be the usual spherical coordinates mapping, given by

$$\mathbf{g} \begin{pmatrix} \rho \\ \phi \\ \theta \end{pmatrix} = \begin{bmatrix} \rho \sin\phi \cos\theta \\ \rho \sin\phi \sin\theta \\ \rho \cos\phi \end{bmatrix}.$$

   Compute $\mathbf{g}^*(dx \wedge dy \wedge dz)$.

8. We say a k-form is *closed* if $d\omega = 0$ and *exact* if $\omega = f\eta$ for some $(k-1)$-form η.
   (a) Prove that an exact form is closed. Is every closed form exact?
   (b) Prove that if ω and ϕ are closed, then $\omega \wedge \phi$ is closed.
   (c) Prove that if ω is exact and ϕ is closed, then $\omega \wedge \phi$ is exact.