MATH 462
Midterm Examination

You will have until Friday, March 7th to complete the following problems. You may use your calculus book, Shifrin's book, your brain, and my brain. Please do not use any other books or anyone else's brain.

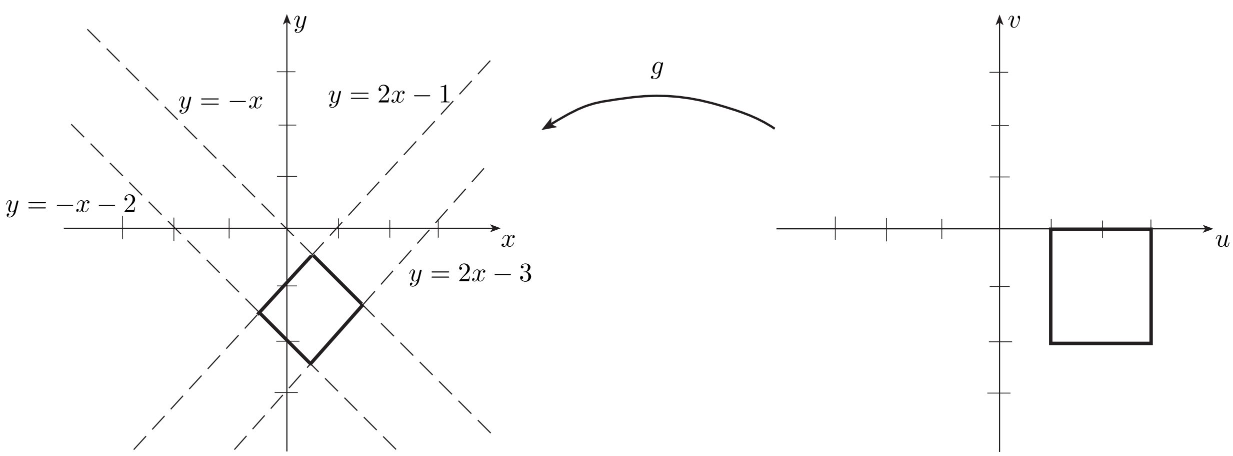
(1) Let $f\begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 + xy - y^2$ and let $P = \{(x, y) \in \mathbb{R}^2 : 1 \le 2x - y \le 3, -2 \le x + y \le 0\}$. Compute the integral

$$\int_P f\begin{pmatrix} x \\ y \end{pmatrix} dA$$

by turning it into an integral of the form

$$\int_\Omega (f \circ g)\begin{pmatrix} u \\ v \end{pmatrix} |\det Dg|\, dA$$

Solution : The first thing is to try to determine what our new coordinates will be. Notice that the region we are integrated over is the space bounded by two sets of parallel lines. A decent choice for the new coordinates will be $u = 2x - y$ and $v = x + y$, as you can see below, this choice of coordinates results in a transformation of a rectangle into our region. Rectangles are typically nice to integrate over.



Now, the function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the transformation that turns the u, v-coordinates into the x, y-coordinates. So, to define $g\begin{pmatrix} u \\ v \end{pmatrix}$ we need to determine how to write x and y in terms of u and v. We begin with the system of equations

$$\left\{\begin{array}{l} u = 2x - y \\ v = x + y \end{array}\right\}$$

and solve for x and y.

$$\left\{\begin{array}{l} u = 2x - y \\ v = x + y \end{array}\right\} \Rightarrow \left\{\begin{array}{l} y = 2x - u \\ v = x + (2x - u) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} y = 2x - u \\ (v + u)/3 = x \end{array}\right\} \Rightarrow \left\{\begin{array}{l} y = 2(v + u)/3 - u \\ (v + u)/3 = x \end{array}\right\}.$$

After "nicefying" this we have an expression for g

$$g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{u+v}{3} \\ \\ \frac{-u+2v}{3} \end{pmatrix}.$$

We will need to compute the Jacobian in order to determine the scaling multiplier. Here we go,

$$Dg = \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

and so $|\det Dg| = |2/9 + 1/9| = 1/3$. All that remains is to compute the composition $(f \circ g)\begin{pmatrix} u \\ v \end{pmatrix}$. We have an expression for x in terms of u and v, and we also have an expression for y in terms of u and v. Indeed, $x = u/3 + v/3$ and $y = -u/3 + 2v/3$. Substituting these into the expression $2x^2 + xy - y^2$ gives us:

$$\begin{aligned}
&2(u/3 + v/3)^2 + (u/3 + v/3)(-u/3 + 2v/3) - (-u/3 + 2v/3)^2 \\
=&2(u^2/9 + 2uv/9 + v^2/9) + (-u^2/9 + 2uv/9 - uv/9 + 2v^2/9) - (u^2/9 - 4uv/9 + 4v^2/9) \\
=&u^2(2/9 - 1/9 - 1/9) + uv(4/9 + uv/9 + 4/9) + v^2(2/9 + 2/9 - 4/9) \\
=&uv.
\end{aligned}$$

If you noticed that $(2x - y)(x + y) = 2x^2 + xy - y^2$ without doing the above calculation, good for you. So, the change of variables theorem tells us

$$\begin{aligned}
\int_{g(\Omega)} f\begin{pmatrix} x \\ y \end{pmatrix} dA_{x,y} &= \int_{\Omega} (f \circ g)\begin{pmatrix} u \\ v \end{pmatrix} |\det Dg| dA_{u,v} \\
&= \int_{\Omega} \frac{uv}{3}\, dA_{u,v}
\end{aligned}$$

Since we have chosen coordinates wisely, the integral over Ω is simply an iterated integral over a rectangle $[1, 3] \times [-2, 0]$.

$$\begin{aligned}
\int_{\Omega} \frac{uv}{3}\, dA_{u,v} &= \int_{-2}^{0} \int_{1}^{3} \frac{uv}{3}\, du\, dv \\
&= \frac{1}{3} \int_{-2}^{0} v \left(\int_{1}^{3} u\, du \right) dv \\
&= \frac{1}{3} \int_{-2}^{0} v \left(\frac{9}{2} - \frac{1}{2} \right) dv \\
&= \frac{4}{3} \int_{-2}^{0} v\, dv \\
&= \frac{4}{3} \left(\frac{0}{2} - \frac{4}{2} \right) \\
&= -\frac{8}{3}.
\end{aligned}$$

(2) Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ in terms of the (nonzero) constants a, b, and c.

Solution : If we let $E = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}$, then we see we wish to compute

$$\int_E 1\, dV.$$

To do this, we will construct a transformation $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that maps the unit ball onto E. Then, by the change of variables formula we will have (where D^3 denotes the unit ball in $\mathbb{R}^3$)

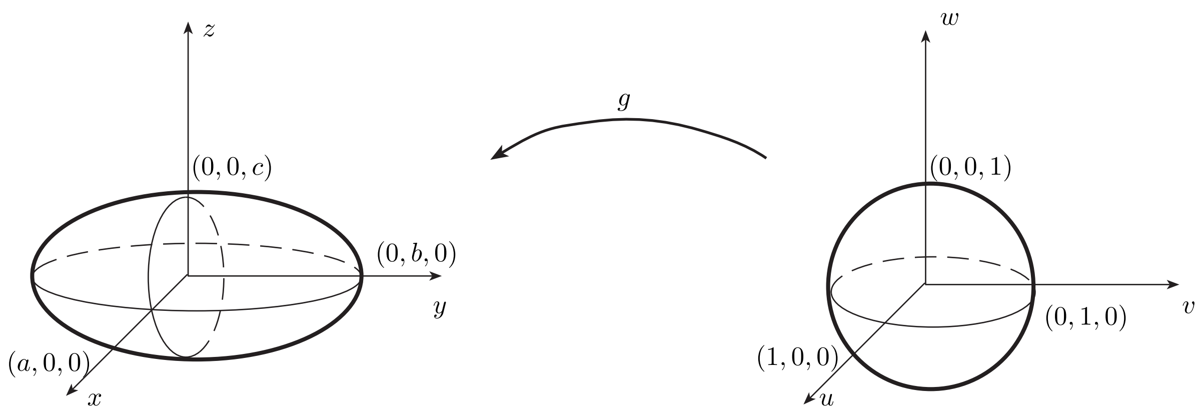
$$\int_E 1\, dV = \int_{D^3} (1 \circ g) \begin{pmatrix} u \\ v \\ w \end{pmatrix} |\det Dg|\, dV_{u,v,w}.$$

Notice that since the integrand of the original integral was just the constant function 1, when we compose this with g we still have the constant function 1. So, our integral gets a little better,

$$\int_{D^3} (1 \circ g) \begin{pmatrix} u \\ v \\ w \end{pmatrix} |\det Dg|\, dV_{u,v,w} = \int_{D^3} |\det Dg|\, dV_{u,v,w}.$$

I claim that the mapping we want is

$$g \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} au \\ bv \\ cw \end{pmatrix}.$$



Observe that each layer of D^3 corresponding to the set of points that satisfies the equation $u^2 + v^2 + w^2 = r^2$, where $r \in [0, 1]$ is mapped to the set of points (au, bv, cw) and, sure enough

$$\frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} + \frac{(cw)^2}{c^2} = u^2 + v^2 + w^2 = r^2$$

which shows that the sphere of radius r is mapped to the ellipse with u, v, and w intercepts ra, rb, and rc respectively. Letting r vary from 0 to 1 shows that g does indeed map the unit ball in $\mathbb{R}^3$ onto the ellipsoid E. The Jacobian of g is given by

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

and so $|\det Dg| = abc$. Thus,

$$\int_{D^3} |\det Dg|\, dV_{u,v,w} = \int_{D^3} abc\, dV_{u,v,w} = abc \int_{D^3} 1\, dV_{u,v,w}.$$

Now we make the observation that $\int_{D^3} 1\, dV_{u,v,w}$ is the volume of the unit sphere in $\mathbb{R}^3$. Since the volume of a sphere is $V = 4/3\pi r^3$ and since $r = 1$ we have,

$$abc \int_{D^3} 1\, dV_{u,v,w} = \frac{4}{3}\pi abc.$$

(3) Find the volume of the "pillow" $\rho = \sin(\theta)$, $0 \le \theta \le \pi$. (Use spherical coordinates).

Solution : Here again we wish to integrate the constant function 1 over the volume bounded by the pillow. That is, we wish to compute

$$\int_P dV.$$

Since the boundary of the region P is the pillow and it is defined in terms of spherical coordinates, it behooves us to transform our coordinates into spherical coordinates. Doing so we get,

$$\begin{aligned}
\int_P dV &= \int_0^{\pi}\int_0^{\pi}\int_0^{\sin\theta} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{\pi}\int_0^{\pi} \sin\phi \left(\int_0^{\sin\theta} \rho^2 \, d\rho\right) d\phi \, d\theta \\
&= \int_0^{\pi}\int_0^{\pi} \sin\phi \left(\left.\frac{\rho^3}{3}\right|_0^{\sin\theta}\right) d\phi \, d\theta \\
&= \int_0^{\pi}\int_0^{\pi} \sin\phi \left(\frac{\sin^3\theta}{3}\right) d\phi \, d\theta \\
&= \frac{1}{3}\int_0^{\pi} \sin^3\theta \left(\int_0^{\pi} \sin\phi \, d\phi\right) d\theta \\
&= \frac{1}{3}\int_0^{\pi} \sin^3\theta \left(-\cos\phi|_0^{\pi}\right) d\theta \\
&= \frac{2}{3}\int_0^{\pi} \sin^3\theta \, d\theta \\
&= \frac{2}{3}\int_0^{\pi} \left(1-\cos^2\theta\right) \sin\theta \, d\theta \\
&= \frac{2}{3}\left[\int_0^{\pi} \sin\theta \, d\theta - \int_0^{\pi} \sin\theta\cos^2\theta \, d\theta\right]
\end{aligned}$$

$$= \frac{2}{3}\left[-\cos\theta|_0^{\pi} + \int_1^{-1} u^2\,du\right]$$

$$= \frac{2}{3}\left[2 + \left.\frac{u^3}{3}\right|_1^{-1}\right]$$

$$= \frac{2}{3}\left[2 - \frac{2}{3}\right]$$

$$= \frac{8}{9}.$$

(4) Find the distance from $(1,1,1)$ to the line of intersection of the planes $2x+y-z=1$ and $x-y+z=2$. [*Hint:* Let (x,y,z) be a point on the line and minimize the square of its distance from $(1,1,1)$.]

Solution : Define the function $f:\mathbb{R}^3 \longrightarrow \mathbb{R}$ to be the square of the distance between the point $(1,1,1)$ and (x,y,z). Then, if we define $g_1(x,y,z) = 2x+y-z-1$ and $g_2(x,y,z) = x-y+z-2$, we see we are trying to minimize the function $f(x,y,z) = (x-1)^2+(y-1)^2+(z-1)^2$ subject to the constraints $g_1(x,y,z)=0$ and $g_2(x,y,z)=0$. The method of LaGrange multipliers tells us we wish to solve the system of equations

$$\begin{cases} \nabla f(x,y,z) = \lambda_1 \nabla g_1(x,y,z) + \lambda_2 \nabla g_2(x,y,z) \\ g_1(x,y,z) = 0 \\ g_2(x,y,z) = 0 \end{cases}$$

Since $\nabla f(x,y,z) = (2(x-1), 2(y-1), 2(z-1))$, $\nabla g_1(x,y,z) = (2,1,-1)$, and $\nabla g_2(x,y,z) = (1,-1,1)$ we can rewrite this system as:

$$\begin{cases} 2x-2 = 2\lambda_1+\lambda_2 \\ 2y-2 = \lambda_1-\lambda_2 \\ 2z-2 = -\lambda_1+\lambda_2 \\ 2x+y-z-1 = 0 \\ x-y+z-2 = 0 \end{cases}$$

After moving everything around and giving some suggestive organizing we have

$$\begin{cases} 2x+0y+0z-2\lambda_1-\lambda_2 = 2 \\ 0x+2y+0z-\lambda_1+\lambda_2 = 2 \\ 0x+0y+2z+\lambda_1-\lambda_2 = 2 \\ 2x+y-z+0\lambda_1+0\lambda_2 = 1 \\ x-y+z+0\lambda_1+0\lambda_2 = 2 \end{cases}$$

So, we wish to row reduce the matrix

$$\begin{bmatrix} 2 & 0 & 0 & -2 & -1 & | & 2 \\ 0 & 2 & 0 & -1 & 1 & | & 2 \\ 0 & 0 & 2 & 1 & -1 & | & 2 \\ 2 & 1 & -1 & 0 & 0 & | & 1 \\ 1 & -1 & 1 & 0 & 0 & | & 2 \end{bmatrix}.$$

The above matrix row reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & 0 & 0 & | & 1/2 \\ 0 & 0 & 1 & 0 & 0 & | & 3/2 \\ 0 & 0 & 0 & 1 & 0 & | & -1/3 \\ 0 & 0 & 0 & 0 & 1 & | & 2/3 \end{bmatrix}$$

and we see that the minimum value occurs when $x=1$, $y=1/2$, and $z=3/2$. (We know this must be a minimum instead of a maximum because there is no maximum distance between a point and a line.) The minimum distance is then

$$\sqrt{f\left(1,\frac{1}{2},\frac{3}{2}\right)} = \sqrt{(1-1)^2+\left(1-\frac{1}{2}\right)^2+\left(1-\frac{3}{2}\right)^2} = \sqrt{\frac{1}{2}}.$$

(5) Let $\Omega \subset \mathbb{R}^n$ be a region. Let $f : \Omega \longrightarrow \mathbb{R}$ and $g : \Omega \longrightarrow \mathbb{R}$ be functions that are integrable on Ω that differ only on the boundary of Ω. Show that

$$\int_\Omega f\,dV = \int_\Omega g\,dV.$$

Solution : It suffices to show

$$\int_\Omega f\,dV - \int_\Omega g\,dV = 0.$$

By extending f and g to functions $\widetilde{f}$ and $\widetilde{g}$ defined on a large rectangle containing Ω so that $\widetilde{f}$ and $\widetilde{g}$ are equal to f and g respectively on Ω and zero elsewhere we realize that we may as well assume that Ω is already a rectangle. Let $h : \Omega \longrightarrow \mathbb{R}$ be defined by

$$h(\mathbf{x}) = \begin{cases} (f-g)(\mathbf{x}) & \mathbf{x} \in \partial\Omega \\ 0 & \text{else} \end{cases}$$

then observe that h is integrable since $h = f - g$. So, by definition, we know that there exists a unique number c so that for any partition $\mathcal{P}$ of Ω

$$L(h, \mathcal{P}) \leq c \leq U(h, \mathcal{P}).$$

Since Ω is a region, we know that its boundary, $\partial\Omega$, must have volume zero. So, given any $\varepsilon > 0$ there exists a union of rectangles R_ε so that $\partial\Omega \subset R_\varepsilon$ and $\text{vol}\,(R_\varepsilon) < \varepsilon$. We can further insist, by throwing out rectangles if necessary, that each rectangle in R_ε actually contains points of $\partial\Omega$ (granted this is a bit silly, but the set of rectangles that contain $\partial\Omega$ don't *necessarily* each need to contain part of $\partial\Omega$). Now suppose that the partition $\mathcal{P}$ is fine enough so that it contains all of the rectangles R_ε.

Lemma 1. *Any of the rectangles in R_ε must contain points inside $\partial\Omega$ as well as points inside $\Omega \setminus \partial\Omega$.*

Proof. We proceed by contradiction. Suppose R is a rectangle in R_ε so that the intersection of R with $\Omega \setminus \partial\Omega$ is empty. This implies that R is a proper subset of $\partial\Omega$ and so $\text{vol}(R_\varepsilon) \geq \text{vol}(R)$. This is contradiction since $\text{vol}(R)$ is a fixed quantity. □

By the lemma, in each R_ε there must be points $\mathbf{x}$ and $\mathbf{y}$ so that $h(\mathbf{x}) = (f-g)(\mathbf{x})$ and $h(\mathbf{y}) = 0$. This tells us that if R' is any rectangle that includes points in $\partial\Omega$, then

$$\inf_{\mathbf{x} \in R'} h(\mathbf{x}) = 0.$$

It follows that $L(h, \mathcal{P}) = 0$.

So, since c is the *unique* constant so that

$$0 = L(h, \mathcal{P}) \leq c \leq U(h, \mathcal{P}),$$

it follows that $c = 0$, and so

$$\int_\Omega h\,dV = 0 \Rightarrow \int_\Omega f\,dV = \int_\Omega g\,dV.$$