MATH 2200 Fall 2002
Homework 10
Selected Solutions

§ 4.2 # 28) Use a linear approximation $L(x)$ to an appropriate function $f(x)$, with an appropriate value of a, to estimate $\sqrt{80}$.

Solution : The linear approximation is going to be a tangent line to the function $f(x) = \sqrt{x}$ at some point $(a, f(a))$. Since the tangent line has the equation

$$L(x) = f'(a)(x - a) + f(a)$$

we had better be able to evalute $f(a)$ as well as $f'(a)$ and we want for a to be as close to 80 as possible. Since,

$$f'(x) = \frac{1}{2\sqrt{x}}$$

we see that if we can compute $\sqrt{a}$ all will be golden. I know that $\sqrt{81} = 9$ and 81 is pretty close to 80, so define $a = 81$. Then the equation of the line becomes,

$$L(x) = \frac{1}{2\sqrt{81}}(x - 81) + \sqrt{81} = \frac{1}{18}(x - 81) + 9 = \frac{1}{18}x + \frac{9}{2}.$$

So, our approximate value from $sqrt 80$ is $L(80) = 80/18 + 9/2 = 161/18 = 8.9\bar{4}$.

§ 4.3 # 22) Determine the intervals on the x-axis on which the given function is increasing as well as those on which it is decreasing.

$$f(x) = x^2 e^{-2x}$$

Solution : We want to know the intervals on which $f'(x)$ is positive and the intervals on which $f'(x)$ is negative.

$$f'(x) = 2xe^{-2x} + x^2(-2)e^{-2x} = 2xe^{-2x}\,(1-x)$$

The derivative is defined everywhere so the only critical points occur when $f'(x) = 0$. Thus the only critical points are $x = 0$ and $x = 1$. So, we need to check the value of the derivative at some point less than zero, some point in between zero and one, and some point greater than one since we know the only place the derivative can possibly change sign is at the critical points.

$$f'(-1) = 2(-1)e^{-2(-1)}\,(1-(-1)) = -4e^2 < 0$$
$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)e^{-2\left(\frac{1}{2}\right)}\left(1-\frac{1}{2}\right) = \frac{1}{2}e^{-1} > 0$$
$$f(2) = 2(2)e^{-2(2)}\,(1-2) = 4e^{-4}(-1) < 0$$

So, we see that $f(x)$ is increasing on the interval $(0,1)$ and it is decreasing on the intervals $(-\infty, 0)$ and $(1, \infty)$.

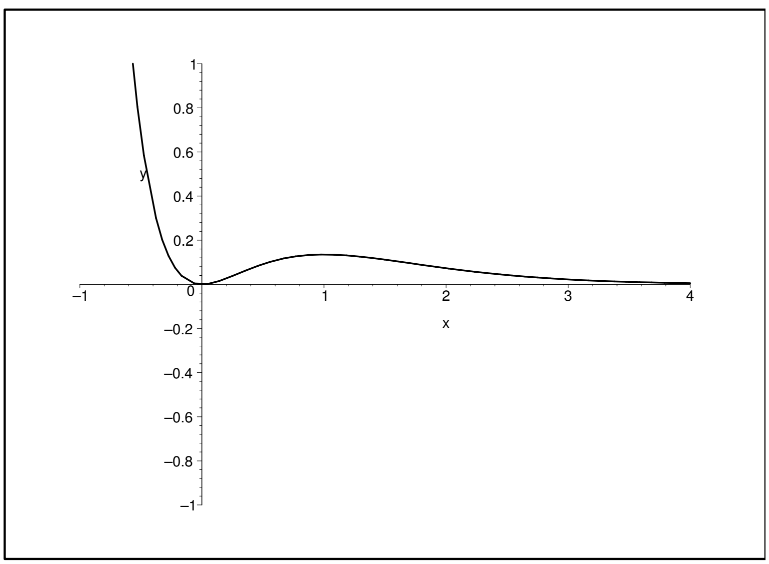

Figure 1: A plot of $f(x) = x^2e^{-2x}$ on the interval $(-1, 4)$.

§ 4.3 # 42) Show that the equation $e^{-x} = x - 1$ has exactly one solution in the interval $[1, 2]$.

Solution : Let $f(x) = e{-}x - x + 1$. Then we are to show that there exists exactly one point c in $[1, 2]$ so that $f(c) = 0$. Note that $f(x)$ is a continuous function on $\mathbb{R}$, so it is continuous on $[1, 2]$. Therefore, by the intermedite value theorem, to show that there exists at least one solution, it suffices to find points a and b in $[1, 2]$ so that $f(a) < 0$ and $f(b) > 0$. Let $a = 2$ and $b = 1$. Then, $f(a) = f(2) = e^{-2} - 2 + 1 = e^{-2} - 1 < 0$, and $f(b) = f(1) = e^{-1} - 1 + 1 = e^{-1} > 0$. So we know that there is at least one solution. To show that there is exactly one solution it suffices to show that the function is either always increasing or always decreasing on the interval $[1, 2]$ since the only way there could be another zero is if the function turned back around to run through the x-axis again thus changing from increasing to decreasin or vice-versa. (Note the last statement is really Rolle's theorem, if there are two points c and d in $[1, 2]$ so that $f(c) = f(d) = 0$, then Rolle's theorem tells us that there must be some point in between c and d where the derivative vanishes. We will show that the derivative is either always positive or always negative, thus it can never vanish and so there can be no d so that $f(d) = 0$). To determine if the funcion is always increasing or always decreasing, we compute the derivative:

$$f'(x) = -e^{-x} - 1.$$

Note that e^{-x} is always positive, so $-e^{-x}$ is always negative. Thus $f'(x) = -e^{-x} - 1$ is always negative and so we see that the function is always decreasing. Thus it can never turn back around to hit the x-axis again.

§ 4.4 # 53) Show that, among all closed cylindrical cans with a given fixed volume, the one with minimal total surface area has height equal to the diameter of its base.

Solution : There are at least two ways to solve this problem. One solution is the brute force technique, while the other involves a more clever approach. Thanks much to those that showed me the clever approach. I will do both starting with the brute force.

We want to show that if h is the height of the cylinder and r is its radius, then $h = 2r$. If the fixed volume is V_0 we have,

$$V_0 = \pi r^2 h$$

and so we know that $h = V_0/(\pi r^2)$. We want to minimize the surface area equation

$$S = 2\pi r h + 2\pi r^2.$$

Using the above information, we can write this as an equation involving one variable.

$$S(r) = 2\pi r \left(\frac{V_0}{\pi r^2} \right) + 2\pi r^2 = \frac{2V_0}{r} + 2\pi r^2.$$

To minimize this function we compute its derivative,

$$S'(r) = -\frac{2V_0}{r^2} + 4\pi r = \frac{4\pi r^3 - 2V_0}{r^2}$$

This isonly undefined at $r = 0$, but this is not in our interval of intersest $(0, \infty)$ so we throw it out. The other critical point occurs when $S'(r) = 0$. So, lets solve for that.

$$S'(r) = 0$$

$$\Rightarrow \frac{4\pi r^3 - 2V_0}{r^2} = 0$$

$$\Rightarrow 4\pi r^3 - 2V_0 = 0$$

$$\Rightarrow r^3 = \frac{2V_0}{4\pi}$$

$$\Rightarrow r = \frac{V_0^{1/3}}{(2\pi)^{1/3}}$$

Ick. Now we need to solve for h and show that $h = 2r$. We know, from above that $h = V_0/(\pi r^2)$. So,

$$\begin{aligned}
h &= \frac{V_0}{\pi\left(\frac{V_0^{1/3}}{(2\pi)^{1/3}}\right)^2} \\
&= \frac{V_0}{\pi\left(\frac{V_0^{2/3}}{(2\pi)^{2/3}}\right)} \\
&= \frac{V_0}{\pi}\left(\frac{(2\pi)^{2/3}}{V_0^{2/3}}\right) \\
&= \left(\frac{V_0}{V_0^{2/3}}\right)\left(\frac{\pi^{2/3}}{\pi}\right)\left(\frac{2^{2/3}}{1}\right) \\
&= \frac{2^{2/3}V_0^{1/3}}{\pi^{1/3}} \\
&= \left(\frac{2}{2}\right)\frac{2^{2/3}V_0^{1/3}}{\pi^{1/3}} \\
&= 2\frac{V_0^{1/3}}{(2\pi)^{1/3}} \\
&= 2r.
\end{aligned}$$

One would hoe that there is an easier way to solve this problem. Sure enough, there is. We can solve this problem using related rates. Suppose that we are changing r and h with respect to time so that the volume remains constant. Then we can write $V_0 = \pi(r(t))^2h(t)$. Then taking a derivative with respect to time we have,

$$0 = 2\pi r(t)r'(t)h(t) + \pi(r(t))^2h'(t) = \pi r(t)\left(2r'(t)h(t) + r(t)h'(t)\right).$$

Since we know that $r(t)$ cannot be zero we know that we can divide both sides of this equation by $\pi r(t)$ obtaining,

$$0 = 2r'(t)h(t) + r(t)h'(t).$$

Furthermore, we know that we want to minimize the surface area which is given by

$$S(t) = 2\pi(r(t))^2 + 2\pi r(t)h(t)$$

and this minimum will occur when $S'(t) = 0$ since there are no boundaries (again we are dealing with an open interval problem). So, we compute the derivaitve,

$$S'(t) = 2\pi(2)r(t)r'(t)+2\pi r'(t)h(t)+2\pi r(t)h'(t) = 2\pi(2r(t)r'(t)+r'(t)h(t)+r(t)h'(t)).$$

We are interested in when $S'(t) = 0$, which occurs when $2r(t)r'(t)+r'(t)h(t)+r(t)h'(t) = 0$ (you can leave off the 2π since it will not affect when $S'(t) = 0$). But recall that we have a clever formulation for 0 from the previous computation $0 = 2r'(t)h(t) + r(t)h'(t)$. So,

$$\begin{aligned}
S'(t) &= 0 \\
\Rightarrow S'(t) &= 2r'(t)h(t) + r(t)h'(t) \\
\Rightarrow 2r(t)r'(t) + r'(t)h(t) + r(t)h'(t) &= 2r'(t)h(t) + r(t)h'(t) \\
\Rightarrow 2r(t)r'(t) + r(t)h'(t) - r(t)h'(t) &= 2r'(t)h(t) - r'(t)h(t) \\
\Rightarrow 2r(t)r'(t) &= r'(t)h(t) \\
\Rightarrow 2r(t) &= h(t) \text{ assuming that } r'(t) \neq 0 \text{ which is a fair assumption.}
\end{aligned}$$

So, we know that at all times the height is exactly twice the radius, that is the height is exactly the diameter as desired.