MATH 2200 Fall 2002
Homework 11
Selected Solutions

§ 4.6 # 30) Find the exact coordinates of the inflecion points and critical points on the graph of the function $f(x) = 3x^5 - 160x^3$.

Solution : We will need the first and second derivatives.

$$f'(x) = 15x^4 - 480x^2 = 15x^2(x^2 - 32) = 15x^2(x - 4\sqrt{2})(x + 4\sqrt{2})$$
$$f''(x) = 60x^3 - 960x = 60x(x^2 - 16) = 60x(x - 4)(x + 4)$$

The critical points occur when $f'(x) = 0$ or when $f'(x)$ is undefined. Since $f'(x)$ is a polynomial, it is defined everywhere. Therefore the only critical points are $x = 0$, $x = 4\sqrt{2}$,and $x = -4\sqrt{2}$.

The inflection points occur when the second derivative changes sign. The only possible places this can occur is when the second derivative is zero or undefined. Thus, the possible inflection points are $x = 0$, $x = 4$, and $x = -4$. We need to do some sign analysis to check to see where (if anywhere) the second derivative changes sign. To do this, we need to examine the sign of $f''(x)$ at some point smaller than -4, some point inbetween -4 and 0, some point in between 0 and 4, and some pont larger than 4.

$$f''(-5) = 60(-5)(-5 - 4)(-5 + 4) = -2700 < 0$$
$$f''(-1) = 60(-1)(-1 - 4)(-1 + 4) = 900 > 0$$
$$f''(1) = 60(1)(1 - 4)(1 + 4) = -900 < 0$$
$$f''(5) = 60(5)(5 - 4)(5 + 4) = 2700 > 0$$

So, the graph of the function $f(x)$ is concave up on the intervals $(-4, 0)$ and $(4, \infty)$. The grpah is concave down on the intervals $(-\infty, -4)$ and $(0, 4)$. The inflection points are then $x = -4$, $x = 0$, and $x = 4$.

The coordinates of the critical points are $(-4\sqrt{2}, 8192\sqrt{2})$, $(0, 0)$, and $(4\sqrt{2}, -8192\sqrt{2})$. The coordinates of the inflection points are $(-4, 7168)$, $(0, 0)$, and $(4, -7168)$.

§ 4.6 # 66) Sketch the graph of the function $f(x) = 3x^5 - 5x^3$ indicating all critical points and inflection points. Apply the second derivative test at each critical point. Show the correct concave structure and indicate the behavior of $f(x)$ as $x \to \pm\infty$.

Solution : Again we will need the first and second derivative.

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2-1) = 15x^2(x-1)(x+1)$$
$$f''(x) = 60x^3 - 30x = 30x(2x^2-1) = 30x(x-1/\sqrt{2})(x+1/\sqrt{2})$$

The function will be increaing when $f'(x) > 0$ and decreasing when $f'(x) < 0$. So, we need to check a point smaller than -1, a point in between -1 and 0, a point in between 0 and 1, and a point larger than 1.

$$f'(-2) = 15(-2)^2(-2-1)(-2+1) = 180 > 0$$
$$f'(-\frac{1}{2}) = 15(-\frac{1}{2})^2(-\frac{1}{2}-1)(-\frac{1}{2}+1) = -\frac{45}{16} < 0$$
$$f'(\frac{1}{2}) = 15(\frac{1}{2})^2(\frac{1}{2}-1)(\frac{1}{2}+1) = -\frac{45}{16} < 0$$
$$f'(2) = 15(2)^2(2-1)(2+1) = 180 > 0$$

So, $f(x)$ is increasing on the intervals $(-\infty,-1)$ and $(1,\infty)$ and $f(x)$ is decreasing on the intervals $(-1,0)$ and $(0,1)$.

Now we need to check concavity.

$$f''(-1) = 30(-1)(2(-1)^2-1) = -30 < 0$$
$$f''(-\frac{1}{2\sqrt{2}}) = 30(-\frac{1}{2\sqrt{2}})(2(-\frac{1}{2\sqrt{2}})^2-1) = \frac{45\sqrt{2}}{8} > 0$$
$$f''(\frac{1}{2\sqrt{2}}) = 30(\frac{1}{2\sqrt{2}})(2(\frac{1}{2\sqrt{2}})^2-1) = -\frac{45\sqrt{2}}{8} < 0$$
$$f''(1) = 30(1)(2(1)^2-1) = 30 > 0$$

So, the graph of $f(x)$ is concave up on the intervals $(-1/\sqrt{2},0)$ and $(1/\sqrt{2},\infty)$. The graph is concave down on the intervals $(-\infty,-1/\sqrt{2})$ and $(0,1/\sqrt{2})$. So, there are inflection points at $x = -1/\sqrt{2}$, $x = 0$, and $x = 1/\sqrt{2}$.

Since the critical point $x = 0$ is also an inflection point, it is neither a local max nor a local min. However, since the critical point $x = -1$ lies withing the interval $(-\infty,-1/\sqrt{2})$ (where the graph of $f(x)$ is concave down) we know that there is a local maximum at $x = -1$ and its value is $f(-1) = 2$. Likewise, the critical point $x = 1$ lies in the interval $(1/sqrt2,\infty)$ (an interval where the graph of $f(x)$ is concave up) and so the point $x = 1$ yields a local minimum value of $f(1) = -2$.

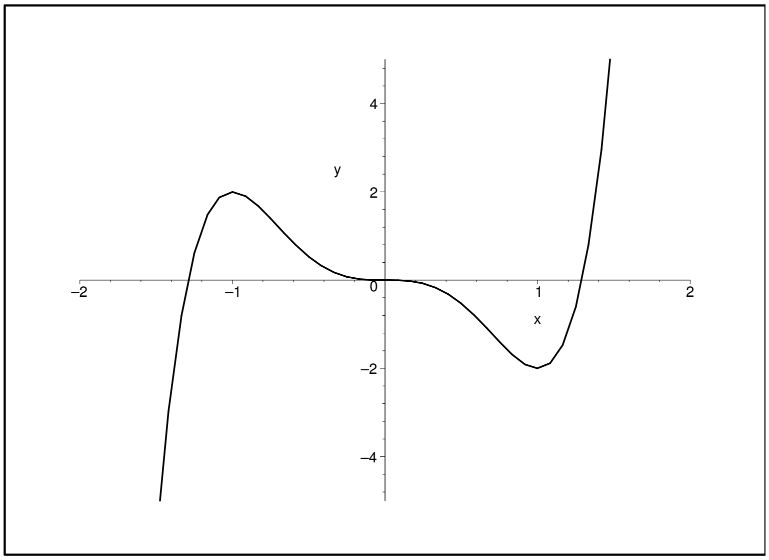

Figure 1: Maple's sketch of the function $f(x) = 3x^5 - 5x^3$.

α) Let

$$f(x) = \frac{x^2 + 2x + 1}{x + 1}.$$

Find a function $g(x)$ so that

$$\lim_{x \to \infty} |f(x) - g(x)| = 0,$$

thereby showing that $f(x)$ is asymptotic to $g(x)$.

Solution : After performing long division, we see that $x + 1$ is a factor of the numerator. That is, the remainder term after performing the long division is zero. So, let $g(x) = x + 1$. Then,

$$\lim_{x \to \infty} |f(x) - g(x)| = \lim_{x \to \infty} \left| \frac{x^2 + 2x + 1}{x + 1} - x + 1 \right|$$

$$= \lim_{x \to \infty} |x + 1 - x + 1|$$

$$= 0$$

So, $f(x)$ is asymptotic to the function $g(x) = x + 1$.