Name:___________________

**Test 3**
Fall 2002
Math 2200 MWF 9:05-9:55am
October 25, 2002

**Directions :** You have 50 minutes to complete all 5 spooky problems on this exam. There are a possible 100 points to be earned on this exam. You may not use programmable calculators; however you may use scientific calculators if you wish. Please be sure to show all pertinent work. *An answer with no work will receive very little credit!* If any portion of the exam is unclear please come to me and I will elaborate provided I can do so without giving away the problem.

1. (20 points)
Show that there exists a function $f(x)$ that is continuous on a closed interval $[a, b]$ that achieves a maximum at a point $c$ in the interval $[a, b]$ where $c \neq a$, $c \neq b$, and $f'(c) \neq 0$.

**Solution :** We know that the maximum must happen either at an endpoint or at a critical point. Recall that a critical point is a point $x$ so that $f'(x) = 0$ *or* $f'(x)$ is undefined. So, we must be looking for an example where the derivative is undefined. My favorite example of such a function is the absolute value function since it has a sharp corner. Sure enough, if I let $f(x) = -|x-1| + 1$, then I see that on the interval $[0, 2]$ $f(x)$ has a maximum of 1 when $x = 1$. But, $1 \neq 0$, $1 \neq 2$, and $f'(1) \neq 0$ (indeed $f'(1)$ is undefined).

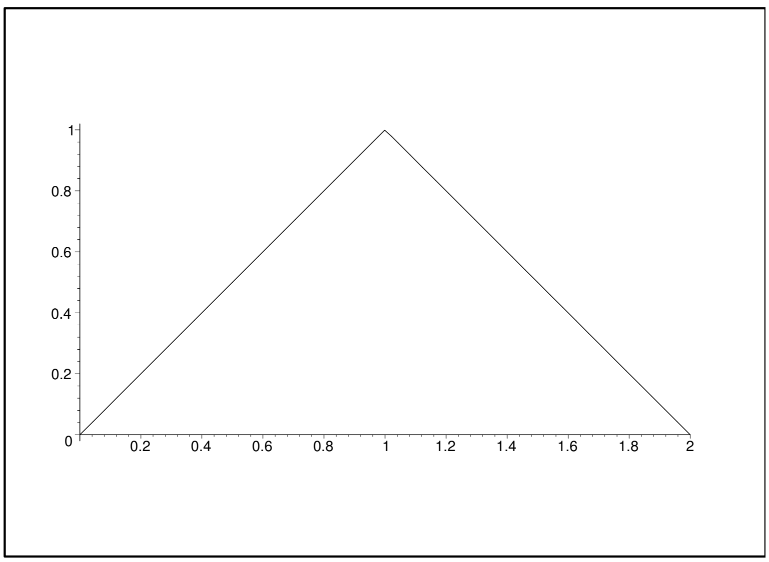


Figure 1: $f(x) = -|x-1| + 1$

2. (20 points)
Compute the derivatives of the following functions with respect to $x$. Continue on the back of this page if you need more room.

(a) $f(x) = e^{1+\ln(x^2+1)}$

(b) $g(x) = \cos(x)\sec(2x)$

(c) $h(x) = e^{\sin(x)}$

(d) $k(x) = \ln(x^2)\sin(\cos(x))$.

**Solution :**

(a) Recall that $e^{a+b} = e^a e^b$ and $e^{\ln(x)} = x$. Using this we can rewrite $f(x)$ so that it looks a little more nice. Indeed,

$$f(x) = e^{1+\ln(x^2+1)} = e^1 e^{\ln(x^2+1)} = e(x^2+1)$$

So, $f'(x) = 2xe$. Or you can do it the gross way using the chain rule.

$$f'(x) = e^{1+\ln(x^2+1)}\left[\left(\frac{1}{x^2+1}\right)(2x)\right]$$

Funny thing is, they are the same.

(b) $g'(x) = -\sin(x)\sec(2x) + \cos(x)\left[2\sec(2x)\tan(2x)\right]$

(c) $h'(x) = e^{\sin(x)}\cos(x)$

(d) $k'(x) = \frac{1}{x^2}(2x)\sin(\cos(x)) + \ln(x^2)\cos(\cos(x))(-\sin(x))$

3. (20 points)
Jack Skellington is trying to construct a fence, using Beastie Blocker$^{TM}$ as the fencing, to hold his gremlins, imps, mephits, and brownies seperately. Given that he has 100ft of Beastie Blocker$^{TM}$ what dimensions will maximize the area he can enclose.

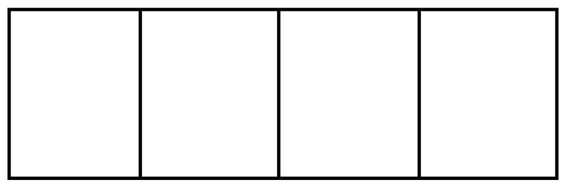

Figure 2: Four regions enclosed with Beastie Blocker$^{TM}$

**Solution :** We are trying to maximize the area. So, if we label the long edge as $y$ and each of the 5 short edges as $x$ we see that we want to maximize the function $A = xy$ subject to the condition $100 = 5x + 2y$. Next we want to write the area function in terms of only one variable. So, we will use the equation about the perimeter to do so. Indeed, $100 = 5x + 2y$ implies that $y = 50 - 5x/2$. Therefore, we can rewrite the area function as $A(x) = x(50 - 5x/2) = 50x - 5x^2/2$. Next we need to find the domain of interest. Indeed, the smallest that $x$ can be is zero so we have the left endpoint. To find the right endpoint we let the other dimension be zero in the constraint equation and solve for $x$. $100 = 5x + 2(0)$ gives us $x = 20$. So, the closed interval of interest is $[0, 20]$. Now we need to search for critical points since $A(x)$ is continuous on $[0, 20]$. $A'(x) = 50 - 5x$ and this is defined everywhere so the only critical point occurs when $A'(x) = 0$. This gives us the critical point $x = 10$ which is in our closed interval. So, now we check to see which one gives us the largest value.

$$A(0) = 0$$
$$A(10) = 500 - 250 = 250$$
$$A(20) = 0$$

So, the largest area occurs when $x = 50\text{ft}$ and $y = 50 - 5(10)/2 = 25\text{ft}$ and the area with these dimensions is $250\text{ft}^2$.

4. (20 points)
Sticky Wickett works in a packaging plant. Recently a horde of vampires came in with $200\pi \text{in}^2$ of material and said,“We need for you to make a closed cylindrical container to hold our supply of blood bleh. It needs to hold a volume of $2000\pi \text{in}^3$ bleh. We'll pick it up tomorrow night bleh.” Prove that Sticky is in a tight spot. (Hint: Given that he has only $200\pi \text{in}^2$ material to work with, how much volume can he enclose?)

**Solution :** As the hint suggests, lets see what possible volume we can attain with the material that was given. The volume of a cylinder is given by the equation $V = \pi r^2 h$ and the surface area of a closed cylinder is $SA = 2\pi r^2 + 2\pi rh$. The surface are is what is being constrained. That is we know $SA = 200\pi = 2\pi r^2 + 2\pi rh$. From this we can write the volume as a function of one variable.

$$\begin{aligned} &200\pi = 2\pi r^2 + 2\pi rh \\ \Rightarrow &\frac{200\pi - 2\pi r^2}{2\pi r} = h \\ \Rightarrow &\frac{100 - r^2}{r} = h \end{aligned}$$

So,

$$\begin{aligned} V(r) &= \pi r^2 \left( \frac{100 - r^2}{r} \right) \\ &= 100\pi r - \pi r^3 \end{aligned}$$

Where the closed interval of interset is $[0, 10]$ (obtained by the physical restrictions $r \geq 0$ and $h \geq 0$ as in the previous problem). Notice that our function is continuous on this interval so we need to just find the critical points.

$$\begin{aligned} &V'(r) = 0 \\ &\Rightarrow 100\pi - 3\pi r^2 = 0 \\ &\Rightarrow 100\pi = 3\pi r^2 \\ &\Rightarrow \frac{100}{3} = r^2 \\ &\Rightarrow \pm\frac{10}{\sqrt{3}} = r \end{aligned}$$

But the point $x = -10/\sqrt{3}$ is outside of our interval, so we can throw it away. So, we need to check the endpoints and our critical point for the maximum.

$$\begin{aligned}
V(0) &= 0 \\
V\left(\frac{10}{\sqrt{3}}\right) &= 100\pi\frac{10}{\sqrt{3}} - \pi\left(\frac{10}{\sqrt{3}}\right)^3 \\
&= \frac{1000\pi}{\sqrt{3}} - \frac{1000\pi}{3\sqrt{3}} \\
&= \frac{3000\pi - 1000\pi}{3\sqrt{3}} \\
&= \frac{2000\pi}{\sqrt{3}} \\
V(10) &= 0
\end{aligned}$$

So, we see that the *maximum* volume he could enclose with the given material is $2000\pi/3\sqrt{3}\text{in}^3$, which is strictly smaller than the ordered $2000\pi\text{in}^3$.

5. (20 points)
Sally is watching as a witch on a broomstick is flying towards her at a constant height of 50ft. She notices that when her line of sight forms an angle of $\pi/4$ radians with the ground that the angle is increasing at a rate of 5 radians per second. What is the witches speed at that instant. (**Bonus :** (5 points) How long will it take for the witch to be directly over Sally?)

**Solution :** First lets draw a picture.

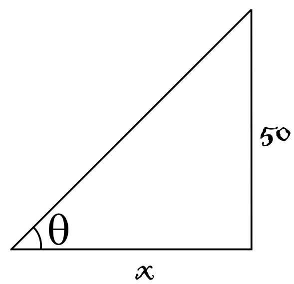


Figure 3: Witch diagram

From the figure we see that we can write $\tan(\theta) = 50/x$, thus $x = 50\cot(\theta)$. So, if we want to find the speed we will need to take the first derivative.

$$\begin{aligned} v &= \frac{dx}{dt} \\ &= 50\left(-\csc^2(\theta)\right)\frac{d\theta}{dt} \end{aligned}$$

So, when we are at the instant when $\theta = \pi/4$ we have

$$\begin{aligned} v &= 50\left(-\left(\frac{1}{\frac{\sqrt{2}}{2}}\right)^2\right)(5) \\ &= 50\left(-\frac{1}{\frac{2}{4}}\right)(5) \\ &= 50(-2)(5) \\ &= -500 \end{aligned}$$

Is this the witch's speed? Heck no. This is the velocity. To find speed we take the absolute value. So, the witch's speed at that instant is exactly 500ft/s.

To find out how long it will take the witch to be directly over Sally we need to just use the $d = rt$ formula. We know that when $\theta = \pi/4$ that

$x = 50 \cot(\theta) = 50$ft. So, if the speed is constant from her on out we see that we have $50 = 500t$ which tells us that the time it will take is exactly $t = 1/10$ seconds. Pretty darn quick witch!